PROJECTIONS IN NORMED LINEAR SPACES

BY

DWIGHT BENJAMIN GOODNER

B.A., William Penn College, 1934
M.A., Haverford College, 1935

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS
IN THE GRADUATE COLLEGE OF THE
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PROJECTIONS IN NORMED LINEAR SPACES*

A normed linear space X has property P_s, $s \geqq 1$, if and only if for every normed linear space Y containing X there is a projection T of norm less than or equal to s of Y onto X. If X has property P_s, X is complete and any space Y which is equivalent to X has property P_s. X has property P_s if and only if for each Banach space Y containing X there is a closed linear subspace G of Y*, the conjugate space of Y, and a projection T′ of Y* onto G such that

a. $\| T' \| \leqq s$

b. $Y = G[\perp] \oplus X$ where $G[\perp] = \{y \mid g(y) = 0,\ y \in Y,\ g \in G\}$

c. $T'h = g$ implies $T'h(g[\perp]+x) = g(g[\perp]+x) = h(x)$, $h \in Y^*$, $g \in G$, $g[\perp] \in G[\perp]$ and $x \in X$.

A new proof is given for Akilov's theorem that if X is of type B_1^+, if the unit sphere C of X has at least upper bound x_0, and if $\| x_0 \| = s$, then X has property P_s.

If X has property P_1 and p and q are distinct extreme points of the unit sphere C of X, then p is at a distance 2 from q and a line segment joining p to q lies wholly on the surface of C. Any point on the surface of C may be joined to either p or -p by a line segment lying entirely on the surface of C. If E is a subset of C and the diameter of E is less than or equal to one, then all faces of C which contain points of E intersect in some point of C.

X does not have property P_1 if there exists in X a hyperplane $H = \{x \mid f(x) = 1 = \| f \|\}$ such that the intersection of H and C is an n-dimensional simplex, $3 \leqq n$. From this it can be shown that no abstract (L)-space (AL) of dimension greater than or equal to 3 has property P_1.

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* The author is greatly indebted to Professor M. M. Day, who directed the preparation of the thesis of which this is an abstract.

If X has property P_1 and an extreme point on its unit sphere C, then X is equivalent to the space of all real-valued continuous functions on some extremally disconnected compact Hausdorff space H. The unit sphere C of X is the closed convex hull of its extreme points. X is reflexive if and only if it is finite dimensional.

If X has an extreme point on its unit sphere, then X has property P_1 if and only if it is equivalent to a space of type B_1^+ whose unit sphere has at least upper bound of norm one.

If an abstract (M)-space has property P_1, it has a unit element. The conjugate space of an abstract (L)-space has property P_1. Known examples of spaces having property P_1 are conjugate spaces of abstract (L)-spaces and hence are all abstract (M)-spaces with unit elements.

VITA

Dwight Benjamin Goodner was born August 15, 1913, at What Cheer, Iowa. He attended high school in Gibson, Iowa, where he graduated in 1930 as co-valedictorian of his class. He entered William Penn College, Oskaloosa, Iowa, in September of 1930 and graduated with honors in 1934, when he received a Bachelor of Arts degree with a major in mathematics. He won the T. Wister Brown scholarship for a year of graduate study at Haverford College, Haverford, Pennsylvania, and was graduated in June of 1935, when he received a Master of Arts degree with a major in mathematics. He then spent the school year of 1935 to 1936 taking special work at the State University of Iowa, Iowa City, Iowa. From September 1936 to June 1937 he was high-school science supervisor at the Iowa Industrial School, Eldora, Iowa. From September of 1937 to July of 1942 he was at South Dakota State College of Agriculture and Mechanic Arts as an instructor and later as an assistant professor in mathematics.

In July of 1942 he entered the United States Naval Reserve as an Ensign. While in the military service, he had sea duty in the Atlantic and South Pacific, and also served as navigation instructor and Educational Services Officer at the USNR Midshipmen's School, Abbott Hall, Northwestern University, Chicago, Illinois. He was a Lieutenant at the time of his release to inactive duty in February of 1946.

In January of 1946 he returned to South Dakota State College and taught mathematics during the winter and spring quarters. In the summer of 1946 he entered the University of Illinois to work for the degree of Doctor of Philosophy. He was a fellow in mathematics from 1948 to 1949. He is a member of Phi Beta Kappa and Phi Kappa Phi.

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